SOLUTIONS OF PROBLEMS.

ALGEBRA.

403. Proposed by C. N. SCHMALL, New York City.

A torpedo-boat 40 miles from shore strikes a rock, making a rent in her hull which admits water at the rate of 15 tons in 48 minutes. The ship's pumps can expel 12 tons in an hour. If 60 tons of water is sufficient to sink the boat, find the average rate of steaming so that it may reach the shore just as it is about to sink.

SOLUTION BY CHRISTIAN HORNUNG, Tiffin, O.

If x represents the rate per hour of steaming, then $40/x$ is the number of hours it takes to reach the shore, and in that time $15 \cdot 60 \cdot 40/48x$ tons of water will have entered the hull, and $12 \cdot 40/x$ tons will have been expelled. Hence $(15 \cdot 60/48) \cdot (40/x) - (12/1) \cdot (40/x) = 60$, and $x = 4\frac{1}{2}$.

Also solved by EMMA GIBSON, F. M. MORGAN, HORACE OLSON, CLIFFORD N. MILLS, WALTER C. ELLS, and J. W. CLAWSON.

404. Proposed by V. M. SPUNAR, Chicago, Illinois.

Show that

$$\begin{aligned}(a+b)(a+b-1)\cdots(a+b-n+1) &= a(a-1)(a-2)\cdots(a-n+1)\\ &+\binom{n}{1}a(a-1)(a-2)\cdots(a-n+2)b+\binom{n}{2}a(a-1)(a-2)\\ &\cdots(a-n+3)b(b-1)+\cdots+b(b-1)(b-2)\cdots(b-n+1).\end{aligned}$$

SOLUTION BY A. M. HARDING, University of Arkansas.

We have

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots;$$

also

$$(1+x)^b = 1 + bx + \frac{b(b-1)}{2!}x^2 + \frac{b(b-1)(b-2)}{3!}x^3 + \cdots.$$

Multiply and equate the coefficients of x^n. Then

$$\binom{a+b}{n} = \binom{a}{n} + \binom{a}{n-1}\binom{b}{1} + \binom{a}{n-2}\binom{b}{2} + \cdots + \binom{b}{n}.$$

If we multiply both members of this equation by n! we obtain the desired result

Also solved by ELIJAH SWIFT, who proved the proposition by induction.

405. Proposed by E. J. MOULTON, Northwestern University.

Given the alternating series

$$S = 1 - 1/2 + 1/3 - 1/4 + 1/5 - \cdots.$$

(*a*) Let S_n be the sum of the first n terms of the series. Show that in order to make the difference $S - S_n$ numerically less than $1/2k$ (k a positive integer) it is necessary and sufficient to make $n = k$; hence S_{500} differs from S by less than .001. (*b*) Let S_n' be the sum of S_n and 1/2 the $(n+1)$th term of the series. Show that the difference $S - S_n'$ is numerically less than $1/2n(n+1)$; hence S_{22}' differs from S by less than .001.

Solution by the Proposer.

Denote the numerical value of the remainder after n terms of the series by $|R_n|$.

(*a*) If we compare the two series

$$|R_{n-1}| = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} + \cdots$$

$$= \frac{1}{n(n+1)} + \frac{1}{(n+2)(n+3)} + \cdots,$$

$$|R_n| = \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \frac{1}{n+4} + \cdots$$

$$= \frac{1}{(n+1)(n+2)} + \frac{1}{(n+3)(n+4)} + \cdots$$

with the series

$$\frac{1}{2n} = \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+4} + \frac{1}{n+4} \cdots\right]$$

$$= \frac{1}{n(n+2)} + \frac{1}{(n+2)(n+4)} + \cdots,$$

we have at once $|R_{n-1}| > 1/2n > |R_n|$, from which the statements follow.

(*b*) From the preceding inequality, we see that

$$\frac{1}{2n} > |R_n| > \frac{1}{2(n+1)};$$

and hence, subtracting $1/2(n+1)$, we have

$$\frac{1}{2n(n+1)} > |R_n| - \frac{1}{2(n+1)} > 0.$$

Letting S be the value of the series, we see

$$S - S_n' = (-1)^n\left[|R_n| - \frac{1}{2(n+1)}\right]$$

and the statements (*b*) follow.

Remark. Perhaps the chief interest of the problem comes from a comparison with a conclusion obtainable immediately from a well-known general theorem on alternating series (Osgood, *Introduction to Infinite Series*, p. 13), namely, that, for our series, to make $|R_n| < .001$ it is *suf-*

ficient to take $n = 1000$; we see that a much smaller number, 500, is also sufficient (and necessary). Part (b) of the problem was originally suggested to the proposer by the fact that when the values of $S_1, S_2, S_3, \cdots$ are plotted on a line, two successive points lie on opposite sides of S and about equally distant from S. The midpoint between S_n and S_{n+1} is S_n'. The value S_n' as an approximation to S may also be obtained as follows: We may write

$$S = S_n + (-1)^n\left[\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots\right]$$
$$= S_n + \frac{(-1)^n}{n+1}\left[\frac{1}{(n+1)(n+2)} + \frac{1}{(n+3)(n+4)} + \cdots\right]$$
$$= S_n + \frac{(-1)^n}{n+1} - (-1)^n\left[\frac{1}{(n+2)(n+3)} + \frac{1}{(n+4)(n+5)} + \cdots\right].$$

Adding the last two equations and dividing by 2, we have

$$S = S_n' + (-1)^n\left[\frac{1}{(n+1)(n+2)(n+3)} + \frac{1}{(n+3)(n+4)(n+5)} + \cdots\right].$$

If we compare this with the second equation we find

$$|S - S_n'| < \frac{1}{n+3}\cdot|S - S_n| \quad\text{or}\quad |S - S_n'| < \frac{1}{2n(n+3)},$$

which is a slightly better test than that given in the problem. In particular it follows that $|S - S_{21}'| < .001$.

407. Proposed by E. B. ESCOTT, University of Michigan.

In computing the values of the natural logarithms of 2, 3, and 5 by the following formulas:

$$\log 2 = 2(7P + 5Q + 3R),$$
$$\log 3 = 2(11P + 8Q + 5R),$$
$$\log 5 = 2(16P + 12Q + 7R),$$

where P, Q, and R are numbers which were computed by infinite series (G. Chrystal, Algebra, Part II, chapt. 28), it is found, on comparing the results with the known values of these logarithms to 15 decimals, that there are the following errors: -2533, -4052, and 6080, respectively. Find the errors in P, Q and R.

SOLUTION BY S. A. JOFFE, New York City.

Denoting the computed values of P, Q and R by the same capital letters with primes, and the errors of computation by the corresponding small letters, we will have: $P' = P + p$; $Q' = Q + q$; $R' = R + r$.

Since $\log 2 = 2(7P + 5Q + 3R) = 2[7(P' - p) + 5(Q' - q) + 3(R' - r)]$, and the computed value $\log' 2 = 2(7P' + 5Q' + 3R')$, we see that the error in $\log' 2$, when compared with the known value of $\log 2$, is $\log' 2 - \log 2$, or

$$2(7p + 5q + 3r) = -2533; \tag{1}$$

similarly:

$$2(11p + 8q + 5r) = -4052, \tag{2}$$

$$2(16p + 12q + 7r) = 6080. \tag{3}$$

This system of simultaneous equations may be solved as follows:

Multiplying (1), (2), and (3) by 4, -1, and -1 respectively, we eliminate q and r, and obtain $2p = -4 \times 2533 + 4052 - 6080$, or $p = -6080$.